COMPUTATION OF THE EFFECTIVE DIFFUSIVITY TENSOR FOR TRANSPORT OF A PASSIVE SCALAR IN A TURBULENT INCOMPRESSIBLE FLOW.

T. KOMOROWSKI AND P. WIDELSKI

Abstract. We consider the passive scalar transport in an incompressible random flow. Our basic result is a proof of the converge of a certain numerical scheme for the computation of the eddy diffusivity tensor. The scheme leads to the formula for the diffusivity expressed in terms of an infinite series. We give a rigorous proof of the geometric bounds on the magnitude of the terms of the series, provided that the Eulerian field is Markovian and Gaussian and its temporal dynamics has a sufficiently large spectral gap. The principal tools used in the proofs are the decomposition of the space of square integrable fields formed over the possible realizations of the Eulerian velocity field in the Gaussian chaos and the hypercontractivity properties of Gaussian measures.

1. Introduction

The transport of a passive scalar field \( T(\cdot, \cdot) \), in a turbulent flow can be modelled by the convection–diffusion equation with a random drift

\[
\begin{align*}
\partial_t T(t, x) + \mathbf{u}(t, x) \cdot \nabla_x T(t, x) &= \kappa \Delta_x T(t, x) \\
T(0, x) &= T_0(x).
\end{align*}
\]

Here \( \mathbf{u} = (u_1, \ldots, u_d) : \mathbb{R} \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d \) is a \( d \)-dimensional, \( d \geq 2 \), random field, usually called the Eulerian velocity, given over a certain probability space \( \mathcal{F} := (\Omega, \mathcal{V}, \mathbb{P}) \) and \( T_0(\cdot) \) is a deterministic initial condition. The drift models the turbulent convection by a flow of a certain fluid. It is therefore assumed to be time–space homogeneous, ergodic, centered and incompressible, i.e. \( \nabla_x \cdot \mathbf{u}(t, x) := \sum_{i=1}^d \partial_{x_i} u_i(t, x) = 0 \). The parameter \( \kappa > 0 \), called the molecular diffusivity, describes the strength of the diffusive dispersion of the medium.

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The passage from the microscopic to macroscopic description of transport is obtained by an appropriate change of scales. For example, under the diffusive scaling the macroscopic coordinates \((t', x')\) are given by \(t' \sim t'/\epsilon^2\), and \(x' \sim x'/\epsilon\), where \(\epsilon \ll 1\) is a certain small parameter. Suppose also that the initial data varies on the macroscopic scale, so it is of the form \(T_0(x)\). Then, in the macroscopic coordinates the rescaled field \(T_\epsilon(t, x) = T(t/\epsilon^2, x/\epsilon)\) (we omit primes here) satisfies

\[
\begin{aligned}
\partial_t T_\epsilon(t, x) + \frac{1}{\epsilon} \mathbf{u} \left( \frac{t}{\epsilon^2}, \frac{x}{\epsilon} \right) : \nabla_x T_\epsilon(t, x) &= \kappa \Delta_x T_\epsilon(t, x) \\
T_\epsilon(0, x) &= T_0(x).
\end{aligned}
\]

One feature that can be proved about the scaled solution, under appropriate assumptions on the statistics of the drift is the self-averaging property of the scaled scalar field. In the weak form it can be stated as follows

\[
\lim_{\epsilon \to 0^+} \left\langle \left[ \int_{\mathbb{R}^d} T_\epsilon(t, x) \phi(x) dx - \int_{\mathbb{R}^d} T^*(t, x) \phi(x) dx \right]^2 \right\rangle = 0, \quad \forall \phi \in C_0^\infty(\mathbb{R}^d).
\]

Here \(\langle \cdot \rangle\) denotes the averaging over the realizations of the drift and \(T^*\) is a (deterministic) solution of a constant coefficient heat equation

\[
\begin{aligned}
\partial_t T^*(t, x) &= \sum_{i,j=1}^d K^*_{i,j} \partial^2_{x_i, x_j} T^*(t, x) \\
T^*(0, x) &= T_0(x).
\end{aligned}
\]

\(K^* = [K^*_{i,j}]\) - the effective diffusivity tensor - is a constant matrix. Since the procedure described above eliminates the inhomogeneity that appears in the convection-diffusion equation on the microscopic level and leads to a space–time homogeneous equation \((1.4)\) it is sometimes referred to as homogenization. To substantiate the averaging property claimed in \((1.3)\) one could first rewrite equation \((1.2)\) in the divergence form. Since the drift \(\mathbf{u}(t, x)\) is of divergence free there exists an anti-symmetric tensor valued potential \(\mathbf{H}(t, x) = [H_{p,q}(t, x)]\), \(H_{p,q}(t, x) = -H_{q,p}(t, x)\) such that \(\mathbf{u}(t, x) = \nabla_x \cdot \mathbf{H}(t, x)\) and the equation in question takes form \(\partial_t T_\epsilon(t, x) - \nabla_x \cdot (a(t/\epsilon^2, x/\epsilon) \nabla_x T_\epsilon(t, x)) = 0\), where the diffusivity matrix \(a(t, x) = [a_{p,q}(t, x)]\) is given by \(a_{p,q}(t, x) = \kappa \delta_{p,q} - H_{p,q}(t, x)\). A rigorous proof of self-averaging for parabolic operators of this form, when the stream matrix \(\mathbf{H}(t, x)\) is \(L^\infty\) bounded, time-space homogeneous and ergodic random field, follows from the results obtained by Zhikov et al. in [12], see Theorem 1, p. 187. However, if \(\mathbf{H}(t, x)\) admits unbounded realizations but possesses an absolute \(p\)-th moment, where \(p > d + 2\), as it is the case in the present paper, then the
averaging in the sense of (1.2) can be concluded from the quenched version of the invariance principle for random characteristics of (1.2) and has been shown in [2]. The aforementioned homogenization results hold both in time dependent and static (time independent) cases. In [7] self-averaging is also shown for Gaussian, time dependent drifts, for which the stream matrix needs not exist. The temporal dynamics of the field is assumed to be Markovian and uniformly mixing on all spatial scales, i.e. possesses a spectral gap, see Theorem 1, p. 528, ibid.

Because the proof of the existence of the effective diffusivity matrix is a result of an application of an appropriate ergodic theorem what is usually left unanswered by the homogenization theorems is how to calculate the effective diffusivity tensor from the statistics of the Eulerian velocity. In this paper we take up a task of providing a formula for computing the effective diffusivity. We consider random drifts that are space-time homogeneous, Gaussian, Markov and whose spectral dynamics possesses sufficiently strong spectral gap.

For the family of the Eulerian velocity fields described above we present a rigorous scheme for calculation of the effective diffusivity matrix that results in an infinite series expansion, see (3.18) below. In addition, we provide a geometric bound for the n–th term of this series, see Theorem 3.1 below, which results in the control of the series tails, see Corollary 3.2. The precise estimate of the size of the spectral gap is possible thanks to formula (3.6).

Another interesting question pertaining to the model with a Gaussian drift is how to relate $K^*$ to the auto-covariance tensor $R(t - s, x - y) := \langle u_i(t, x)u_j(s, y) \rangle$ that, as it is well known, characterizes the Gaussian drift $u(\cdot, \cdot)$. To simplify our calculations we suppose further that the field is spatially isotropic and its spectral gap is identical for all spatial scales. The specific form of the spectrum of the co-variance matrix is presented in (2.1) below. However, as it becomes apparent during the course of the argument, our proofs do not depend on isotropy of the drift. We could also admit fields whose mixing rate vary on different scales, so long as it is bounded away from zero by a certain constant that is not too small. In Section 4 we give a formula for the n-th term of the series for the effective diffusivity in terms of the spectrum of the velocity field, see (4.10). Due to the fact that this term is
in principle, i.e. discarding some possible cancellations, a sum of \( n! \) terms the computational value of this formula would be severely limited. Thanks to Theorem 3.1 however we are able to control the size of a particular term of the series.

Let us describe briefly the main ideas used in the derivation of the formula for the effective diffusivity. The computation is contingent on finding, in an appropriate space, the solution of the cell problem \((\ref{51002})\), in the literature of the subject it is called the corrector field. In Section 3 below we propose a numerical scheme for computing this field, see the formulas \((\ref{41201})\) and \((\ref{41202})\) for the definition of the scheme. The corrector field is then given by \((\ref{42806})\) and the eddy diffusivity can be calculated using \((\ref{90902})\). To gain appropriate estimates on the \(L^2\)-norm of the \( n \)-th term \( \psi_n \) appearing in the scheme, see \((\ref{91001})\) for its definition, we use the decomposition of the space of square integrable functionals formed over the Eulerian velocity field at the given snap-shot of time (say when \( t = 0 \)) into the Gaussian chaos. This together with the hypercontractivity property of Gaussian measure and spectral gap estimate for the dynamics of the Eulerian field produce geometric bound for the respective term, see Proposition 3.1 for the precise statement of the bounds.

Finally, in Section 4 we provide an explicit formula for the \( n \)-th term of the series \((\ref{90902})\), see Proposition 4.4. The formula is stated using the language of Feynman graphs. As we have already mentioned before, in practical calculations, this formula should be coupled with the estimate \((\ref{90903})\) on the tails of the series expansion for the effective diffusivity.

2. The description of the model

2.1. Homogeneous, Gaussian random drifts. We suppose that:

V 1) \( \mathbf{u} : \mathbb{R} \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d \) is a zero mean, space-time homogeneous, spatially isotropic, Gaussian random field over the probability space \( \mathcal{T} \)

V 2) the auto-covariance matrix of the field, cf. \((\ref{1.5})\), is given by

\[
R(t-s, x-y) = b \int_{\mathbb{R}^d} \cos((x-y) \cdot k) e^{-a|t-s|} \mathcal{E}(|k|)|k|^{1-d} \hat{\Gamma}(k) \, dk.
\]

Here \( a, b > 0, \hat{\Gamma}(k) = [\hat{\Gamma}_{i,j}(k)] \), with \( \hat{\Gamma}_{i,j}(k) := \delta_{i,j} - k_i k_j |k|^{-2} \). We assume that the power energy spectrum satisfies the power law

\[
\mathcal{E}(k) := 1_{[0,K_0]}(k) k^{1-2\alpha},
\]
where $K_0 > 0$ is fixed and $\alpha < 1$, to ensure the $L^2$-integrability of the field.

**Remark 2.1.** It is well known that, thanks to (2.2), such a random field possesses a modification that is $\mathbb{P}$ a.s. jointly locally Hölder continuous in $(t, x) \in \mathbb{R} \times \mathbb{R}^d$ and $C^\infty$ smooth in $x$ for any fixed $t \in \mathbb{R}$.

**Remark 2.2.** A direct calculation yields

\[
\langle |u(0, 0)|^2 \rangle = \frac{b}{2} (d-1) \omega_{d-1} \int_0^{K_0} \frac{dk}{k^{2\alpha-1}} = \frac{b}{2} (d-1) \omega_{d-1} K_0^{2(1-\alpha)} (1-\alpha)^{-1},
\]

where $\omega_{d-1}$ denotes the surface measure of $S^{d-1}$ – the unit sphere in $\mathbb{R}^d$. □

**Remark 2.3.** The presence of the factor $\tilde{\Gamma}(\cdot)$ in the formula assures that the spatial realizations of the field are incompressible. The parameter $a > 0$, called the spectral gap, controls the rate at which the field decorrelates in the temporal variable. In our model we assume that this rate is constant on all spatial scales. Let us mention here one particular case that has been widely studied in the literature. When $a = b$ and $a \to +\infty$ the auto-covariance matrix convergence (in the distribution sense) to the auto-covariance matrix of a $\delta$–correlated velocity field, the so-called Kraichnan model, see [kraich], given by

\[
R(t - s, x - y) = \delta(t - s) \int_{\mathbb{R}^d} \cos((x - y) \cdot k) \mathcal{E}(|k|)|k|^{1-d} \tilde{\Gamma}(k) \, dk.
\]

The effective diffusivity in this case is given explicitly and equals

\[
\kappa_* = \kappa + \frac{1}{2} \int_{|k| \leq K_0} \mathcal{E}(|k|) |k|^{1-d} \tilde{\Gamma}_1(k) dk = \kappa + \frac{1}{4} \omega_{d-1} \left( 1 - \frac{1}{d} \right) K_0^{2(1-\alpha)} (1-\alpha)^{-1}.
\]

□

2.2. Formula for the effective diffusivity. An abstract cell problem. Thanks to isotropy of the Eulerian velocity field $u(\cdot, \cdot)$ the effective diffusivity tensor must commute with any rotation. Hence it is of the form

\[
K^* = \kappa_* I, \quad \text{where} \quad \kappa_* = \kappa + d_*
\]

and $d_*$ is called eddy diffusivity. In order to determine eddy diffusivity one needs to solve an auxiliary cell problem for the corrector, see e.g. [kom][7] (4.16) p. 537. To formulate this problem
we need to introduce an appropriate functional space that is big enough to contain all possible spatial realizations of the velocity field at any given time instant.

Suppose that $m$ is a positive integer and $\varrho(x) := (1 + |x|^2)^{-\varrho}$, $x \in \mathbb{R}^d$, where $\varrho > d/2$. Let $H$ be the Hilbert space of $d$-dimensional incompressible vector fields that is the completion of $C_{0,\text{div}}^\infty := \{ f \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^d) : \nabla \cdot f = 0 \}$ with respect to the norm

$$||f||_H^2 := \int_{\mathbb{R}^d} (|f(x)|^2 + |\nabla_x f(x)|^2 + \cdots + |\nabla_x^m f(x)|^2) \varrho(x) \, dx.$$ 

We can always assume that $m$ is big enough (e.g. $m > d/2 + 1$) so, thanks to the Sobolev embedding, any $f \in H$ is of $C^1$ class of regularity. The presence of a weight $\varrho(x)$ follows from the fact that for a given $t$ the spatial realizations of the Gaussian field $u(t, \cdot)$ grow, as $C \log \frac{1}{|x|}$, for $|x| \gg 1$, see e.g. [11].

Let $\mu$ be the law in $H$ of the Gaussian velocity field $u(0, \cdot)$. Denote $L^2 := L^2(\mu)$, $L^2_0$ its subspace consisting of $F$ such that $\int_F \, d\mu = 0$. The measure $\mu$ is Gaussian of zero mean, i.e. $\int f(0) \mu(df) = 0$, with auto-covariance

$$\int f(x) \otimes f(y) \mu(df) = b \int \cos((x - y) \cdot k) \frac{\mathcal{E}(|k|)}{|k|^{d-1}} \tilde{\Gamma}(k) \, dk. \tag{2.6}$$

For any two vectors $a = (a_1, \ldots, a_d)$, $b = (b_1, \ldots, b_d)$ symbol $a \otimes b$ denotes a $d \times d$ matrix $[a_i b_j]$. (2.6) implies homogeneity of $\mu$, i.e. $\mu \tau_x = \mu$, $\forall x \in \mathbb{R}^d$, where $\tau_x : H \to H$ is given by $\tau_x f(\cdot) := f(x + \cdot)$. We define the abstract spatial gradient operator $\nabla$ as follows. Let

$$D_p F(f) := \partial_{x_p|\cdot} F(\tau_x(f)), \quad p = 1, \ldots, d \tag{2.7}$$

for $F \in L^2$, such that all the partial derivatives on the right hand side of (2.7) exist in the $L^2$ sense. Let $\nabla F := (D_1 F, \ldots, D_d F)$. The abstract Laplace operator $\Delta$ can be defined as

$$\Delta F = \sum_{p=1}^d D_p^2 F,$$

for those $F$ for which the second partials exist in the $L^2$ sense.

**Remark 2.4.** Observe that, due to incompressibility of $u$, we have

$$\langle \tilde{u} \cdot \nabla F, G \rangle_{L^2} = -\langle \tilde{u} \cdot \nabla G, F \rangle_{L^2}. \tag{2.8}$$
Here
\[
\tilde{u}(f) = (\tilde{u}_1(f), \ldots, \tilde{u}_d(f)) := f(0)
\]

Remark 2.5. The operator \( \kappa \Delta \) is the generator of a semigroup of symmetric Markov contractions on \( L^2 \), given by
\[
S(t)F(f) = \int_{\mathbb{R}^d} r_\kappa(t, y) F(\tau_y f) dy, \quad F \in L^2,
\]
where
\[
r_\kappa(t, x) := \frac{1}{(4\kappa \pi t)^{d/2}} \exp \left\{ -\frac{|x|^2}{4\kappa t} \right\}.
\]
It is therefore a self-adjoint, negative definite operator.

To introduce the temporal derivative \( D_0 F \) we need to describe in more details the dynamics of the \( \mathcal{H} \)-valued stochastic process \((u(t, \cdot))_{t \geq 0}\). This process can be thought of as the time stationary solution of an \( \mathcal{H} \)-valued linear stochastic differential equation
\[
du(t) = -au(t) dt + \sqrt{2a} B dW(t),
\]
with \( u(0) = u(0, \cdot) \). Here \( W(\cdot) \) is a cylindrical Wiener process on \( L^2_{div}(\mathbb{R}^d, \mathbb{R}^d) \) - the space of all square integrable, incompressible \( d \)-dimensional vector fields - defined over the probability space \( T \) and \( B : L^2_{div}(\mathbb{R}^d, \mathbb{R}^d) \to \mathcal{H} \) is a Hilbert–Schmidt operator given by
\[
\tilde{B} \psi(k) = \sqrt{E(|k|)} |k|^{(1-d)/2} \hat{\psi}(k).
\]
Both here and below \( \hat{\psi} \) denotes the Fourier transform of a given function \( \psi \). We refer a reader to consult \( \text{kom}[7] \) for details of this construction.

The Eulerian velocity field \( u(t, x) \) can be then identified with \( u(t)(x) \). Let \( D_0 : D(D_0) \to L^2 \) be the generator of \( u(\cdot) \) and \( (P(t))_{t \geq 0} \) the corresponding semigroup of Markov operators. It can easily be shown that the dynamics described by (2.12) and (2.13) is reversible, i.e. each \( P(t) \) is self-adjoint, see (3.4) p. 530 of \( \text{kom}[7] \).

The abstract cell problem can be formulated as follows
\[
\kappa \Delta \chi + D_0 \chi - \tilde{u} \cdot \nabla \chi = \tilde{u},
\]
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see (4.16) p. 537 of ibid. Here \( \tilde{u} \) is given by (4.16) and \( \tilde{u} \) denotes one of the components of \( \tilde{u} \). To fix our attention we shall admit \( \tilde{u} := \tilde{u}_1 \). The unknown field \( \chi \) is called a corrector. Thanks to the assumed isotropy of the field the eddy diffusivity \( d_* \) is given by

\[
d_* := - (\tilde{u}, \chi)_{L^2},
\]

see (5.10) p. 544 of [7]. Unfortunately, the equation (5.10) needs not have a solution in \( L^2 \). This is, for instance, a typical situation in the case of static fields. Then, one can only guarantee the existence of \( \chi \) in a distribution sense. However, under the assumption that the dynamics of \( u(\cdot) \) possesses a spectral gap, i.e. its spectral measure is of the form (2.17) with \( a > 0 \), one can show that there exists a unique \( \chi \in L^2 \) solving (2.17) and satisfying \( \int \chi d\mu = 0 \), see (4.16) p. 537 of ibid.

It is immensely difficult in general to perform explicit calculations of eddy diffusivity with the help of formula (2.15). This is due to the fact that the cell problem (2.14) is formulated for functionals defined over an infinite dimensional Hilbert space and it is very seldom possible to solve it explicitly. However, in the Gaussian case it is possible to give a numerical scheme for calculating eddy diffusivity with the help of a decomposition of \( L^2 \) into the Gaussian chaos.

2.3. The decomposition of \( L^2 \) using Gaussian chaos. Let \( P_n \) be the \( L^2 \) closure of the linear space spanned by the monomials \( f \mapsto \langle \varphi_1, f \rangle \ldots \langle \varphi_m, f \rangle \), where \( m \leq n, \varphi_1, \ldots, \varphi_m \in S_d \). Here \( S_d \) is the space of incompressible fields belonging to the Schwartz class and \( \langle g, f \rangle := \int_{\mathbb{R}^d} g(x) \cdot f(x) dx \) for any \( f \in S_d, g \in \mathcal{H} \). \( H_n := P_n \ominus P_{n-1} \) is called the space of \( n \)-th degree Hermite polynomials. We denote by \( \Pi_n \) the orthogonal projection of \( L^2 \) onto \( H_n \). It is well known, see e.g. Theorem 2.6 p. 16 of [6] that \( L^2 = \bigoplus_{n \geq 0} H_n \). Thanks to the fact that the group induced by shift \( U^x F := F(x) \) leaves each \( H_n \) invariant we see from (2.10) that \( S(t)(H_n) = H_n \) for each \( n \).

The Hermite polynomials provide also for a neat description of the operator \( D_0 \). Namely, for any \( F \in H_n \) we have \( F \in D(D_0) \) and

\[
D_0 F = -anF.
\]

(2.16)

(2.16) can be seen by using formula for the generator of \( (u(t, \cdot))_{t \geq 0} \) contained on p. 207 of [9]. A simple consequence of (2.16) is the following spectral gap estimate of \( D_0 \)

\[
-(D_0 F, F)_{L^2} \geq a\|F\|_{L^2}^2, \quad \text{for any } F \in L^2 \text{ s.t. } \int F d\mu = 0.
\]

(2.17)
Another consequence of \((2.16)\) and the fact that each \(H_n\) is invariant under semigroup \(S(\cdot)\) is commutation relation \(P(t)S(s) - S(s)P(t) = 0\) for arbitrary \(t, s \geq 0\). Hence \(R(t) := S(t)P(t)\), \(t \geq 0\) defines a semigroup of self-adjoint operators, whose generator equals \(\kappa \Delta + D_0\) on a sufficiently large subspace \(\mathcal{C}\) that constitutes a core for both \(\kappa \Delta, D_0\). One can for instance take

\[
\mathcal{C} := \bigcap_{p \in (2, \infty)} W^{2,p} \cap D(D_0),
\]

where \(W^{m,p}\) denotes the closure of those \(F\), for which \(x \mapsto F\tau_x\) possesses \(m\) derivatives at \(0\) in the norm given by \(\|F\|_{m,p} := \sum_{i_1 + \ldots + i_d \leq m} \|D_{i_1} \ldots D_{i_d} F\|_{L^p}\).

2.4. The integral representation of the corrector.

Let \(LF := \kappa \Delta F + D_0 F - \tilde{u} \cdot \nabla F, F \in \mathcal{C}\). It can be shown, see Proposition 3 p. 536 of [3], that the closure of \(\mathcal{L}\) is a generator of a semigroup \((Q(t))_{t \geq 0}\) of Markov operators on \(L^2\) that leaves \(\mathcal{C}\) invariant. This semigroup is exponentially stable with \(\|Q(t)F\|_{L^2} \leq e^{-\alpha t} \|F\|_{L^2}\) for any \(F\) satisfying \(\int Fd\mu = 0\), see (4.5) p. 536 ibid. Using the semigroup we can write the unique zero mean solution to the Poisson equation \((2.14)\), cf. (4.16) p. 537 of [7]

\[
\chi := - \int_0^\infty Q(t) \tilde{u} dt.
\]

Since \(\tilde{u} \in \mathcal{C}\) we conclude that also \(\chi \in \mathcal{C}\). From \((2.14)\) and the spectral gap estimate \((2.17)\) we can also infer that for a non-trivial Eulerian velocity field

\[
d_* = -(S\chi, \chi)_{L^2} = \kappa \|\nabla \chi\|^2_{L^2} - (D_0 \chi, \chi)_{L^2} > \kappa \|\nabla \chi\|^2_{L^2} > 0
\]

and in consequence \(\kappa_* > \kappa\). The latter inequality highlights a well known physical fact that in an incompressible medium a turbulent convection enhances diffusive properties of the medium.

Remark 2.6. Note that in our case ”the time derivation operator” \(D_0\) is self-adjoint and satisfies the spectral gap estimate \((2.17)\). A differentiation with respect to time operator \(D_0\) can be in fact introduced for environments that are only stationary in \(t\) (with no additional assumption on the structure of their temporal dynamics), cf. par. 3, p. 193 of [12]. However, when \(D_0\) is defined in such a general way discarding additional information about the dissipative properties of the evolution of the environment in time it is an anti-self-adjoint operator (conditioning on the environment up to a given moment of time becomes averaging w.r.t. a trivial, deterministic, probability measure) and the spectral gap inequality \((2.17)\) fails to be...
true. In fact, then \((D_0F,F)_{L^2} = 0\) for all \(F\). In consequence, the definition of \(\chi\) via (2.18) would not make sense in \(L^2\) under such general circumstances but one can make sense of it in an appropriate distribution space, see par. 4, p. 194 of ibid. □

3. Numerical scheme for calculation of the effective diffusivity

Since \(\tilde{u} \in \mathcal{P}_1\), \(\int \tilde{u} d\mu = 0\) and the spectrum of \(\kappa\Delta + D_0\) restricted to \(L^2_0\) has a gap of size at least \(a > 0\) we can find a unique \(\psi_1 \in \mathcal{P}_1\) that is the solution of

\[
(\kappa\Delta + D_0)\psi_1 = \tilde{u}
\]

and satisfies \(\int \psi_1 d\mu = 0\). In fact

\[
\psi_1 := -\int_0^\infty R(t)\tilde{u} dt.
\]

Suppose we have already defined \(\psi_n \in \mathcal{P}_n\), \(n = 1, \ldots, N\) for a certain \(N\). Set

\[
u_N := \tilde{u} \cdot \nabla \psi_N \in \mathcal{P}_{N+1}.
\]

Note that

\[
\int \nu_N d\mu = \int \nabla \cdot (\tilde{u} \psi_N) d\mu = -\int \nabla 1 \cdot \tilde{u} \psi_N d\mu = 0,
\]

so

\[
\psi_{N+1} := -\int_0^\infty R(t)\nu_N dt \in \mathcal{P}_{N+1}
\]

is the unique zero mean solution of

\[
(D_0 + \kappa\Delta)\psi_{N+1} = \nu_N.
\]

**Theorem 3.1.** Let

\[
q := \left(\frac{6}{a\kappa}\right)^{1/2} \|\tilde{u}\|_{L^2}.
\]

Then,

\[
\|\psi_N\|_{L^2} \leq q^{N-1}\|\psi_1\|_{L^2}, \quad \forall N \geq 1.
\]

**Proof.** Denote \(\psi_{N,k} := \Pi_k \psi_N\) (the projection onto the space of \(k\)-th degree Hermite polynomials) and

\[
\|F\|_{L^2}^2 := \sum_{k=1}^{+\infty} k \|\Pi_k F\|_{L^2}^2.
\]
Obviously $\psi_{N,k} = 0$ for $k > N$. Projecting both sides of (3.3) onto $H_k$ and taking the scalar product against $\psi_{N+1,k}$ we obtain

\[
ak \| \psi_{N+1,k} \|_{L^2}^2 + \kappa \| \nabla \psi_{N+1} \|^2_{L^2} = -(\tilde{u} \cdot \nabla \psi_{N,k+1}, \psi_{N+1,k})_{L^2} - (\tilde{u} \cdot \nabla \psi_{N,k-1}, \psi_{N+1,k})_{L^2}.
\]

The last equality is a consequence of the following two observations. First, note that $\Pi_k (\tilde{u} \cdot \nabla \psi_{N,l}) \neq 0$ only when $l = k - 1, k, k + 1$. In addition, thanks to (2.8),

\[
(\Pi_k (\tilde{u} \cdot \nabla \psi_{N,k}), \psi_{N,k})_{L^2} = (\tilde{u} \cdot \nabla \psi_{N,k}, \psi_{N,k})_{L^2} = 0.
\]

To estimate the right hand side of (3.8) we consider two cases. First, when $k = 1$ then the second term on the right hand side vanishes. On the other hand, thanks to incompressibility of $u(\cdot, \cdot)$, the first term equals $(\tilde{u} \cdot \nabla \psi_{N+1,1}, \psi_{N,2})_{L^2}$. Its absolute value is, by virtue of Cauchy-Schwartz inequality, less than or equal to

\[
\| \psi_{N,2} \|_{L^2} \left[ \sum_{i,j=1}^d \left( \int (D_j \psi_{N+1,1})^2 \right)^{1/2} \right]^{1/2}
\]

\[
\leq \| \psi_{N,2} \|_{L^2} \left[ \sum_{i,j=1}^d \left( \int (D_j \psi_{N+1,1})^4 \right)^{1/2} \right]^{1/2}
\]

\[
\leq \sqrt{3} \| \psi_{N,2} \|_{L^2} \| u \|_{L^2} \| \nabla \psi_{N+1,1} \|_{L^2}
\]

In the last inequality we used the fact that for any centered Gaussian random variable $X$ its fourth moment $E X^4 = 3(E X^2)^2$.

For $k \geq 2$ we estimate as follows. The first term on the right hand side of (3.8) equals $(\tilde{u} \cdot \nabla \psi_{N+1,k}, \psi_{N,k+1})_{L^2}$ and its absolute value can be estimated from above by

\[
\| \nabla \psi_{N+1,k} \|_{L^2} \left[ \int (|\tilde{u}|^2)^{1/2} \right]^{1/2}
\]

\[
\leq \sqrt{d} \| \nabla \psi_{N+1,k} \|_{L^2} \| \tilde{u} \|_{L^2} \| \psi_{N,k+1} \|_{L^2} \| \psi_{N,k+1} \|_{L^2} (m - 1)
\]

for an arbitrary $m > 1$. The hypercontractivity property of Gaussian measures on $L^p$ spaces, see Theorem 5.10 p. 62 in [Janson], allows us to estimate

\[
\| \psi_{N,k+1} \|_{L^2} (m - 1) \leq \left( \frac{2m}{m - 1} - 1 \right)^{(k+1)/2} \| \psi_{N,k+1} \|_{L^2}
\]
\[ \frac{(m + 1)}{(m - 1)} \frac{(k+1)^2}{m} \| \psi_{N,k+1} \|_{L^2}. \]

On the other hand, \( \tilde{u} \) is a Gaussian r.v. under \( \mu \) hence

\[
(3.11) \quad \| \tilde{u} \|_{L^{2m}} = [(2m - 1)!2/(2m)]^{1/(2m)} \| \tilde{u} \|_{L^2}.
\]

Here \((2m - 1)!2 = 1 \cdot 3 \cdot \ldots (2m - 1)\). Using Stirling’s formula, see e.g. \([406]\) of \([5]\), \( m! = \sqrt{2\pi m}(ne^{-1})^m e^{\theta/(12m)} \) for some \( \theta \in (0, 1) \) we conclude from \((3.11)\) that

\[
(3.12) \quad (2m - 1)!2^{1/(4m)} \left(1 - \frac{1}{2m - 1}\right)^{1/(4m)} \exp \left\{ [24m(2m - 1)]^{-1} \right\}. \]

Summarizing, from \((3.9)-(3.12)\) we conclude that

\[
(3.13) \quad |(\tilde{u} \cdot \nabla \psi_{N,k+1}, \psi_{N,1,k})_{L^2}| \leq C \left( \frac{k + 1}{2}, m \right) \| \tilde{u} \|_{L^2}(k + 1)^{1/2} \| \psi_{N,k+1} \|_{L^2} \| \nabla \psi_{N,1,k} \|_{L^2}
\]

\[
\leq \frac{1}{2\kappa} C^2 \left( \frac{k + 1}{2}, m \right) \| \tilde{u} \|_{L^2}(k + 1)^{1/2} \| \psi_{N,k+1} \|_{L^2} + \frac{\kappa}{2} \| \nabla \psi_{N,1,k} \|_{L^2}^2.
\]

Here

\[
(3.14) \quad C(p, m) := (2\pi)^{-1/2}(2m - 1)(m - 1)^{-1/2} \left( \frac{m + 1}{m - 1} \right)^{p/2}
\times 2^{1/(4m)} \left(1 - \frac{1}{2m - 1}\right)^{1/(4m)} \exp \left\{ [24m(2m - 1)]^{-1} \right\}.
\]

Likewise,

\[
(3.15) \quad |(\tilde{u} \cdot \nabla \psi_{N,k-1}, \psi_{N,1,k})_{L^2}| \leq \frac{1}{2\kappa} C^2 \left( \frac{k - 1}{2}, m' \right) \| \tilde{u} \|_{L^2}(k - 1)^{1/2} \| \psi_{N,k-1} \|_{L^2} + \frac{\kappa}{2} \| \nabla \psi_{N,1,k} \|_{L^2}^2.
\]

Note that for \( m = 2k + 1 \) we have

\[
C'((k + 1)/2, 2k + 1) \leq 2^{1/12} e^{-1/2 + 1/360} \left(2 + \frac{1}{2k}\right) \left(1 + \frac{1}{k}\right)^{k/2} < 1.0595 \left(2 + \frac{1}{2k}\right)
\]

hence \( C^2((k + 1)/2, 2k + 1) < 6 \) for \( k \geq 2 \). A direct calculation also shows that \( C^2(1, 3) \approx 5.1620 \). Likewise, for \( m' = 2k - 1 \) we have \( C^2((k - 1)/2, 2k - 1) < 6, \ k \geq 2 \. Summing up \((3.8)\) over \( k \) and using \((3.13)-(3.15)\) we conclude therefore that

\[
(3.16) \quad a \sum_{k=1}^{N+1} k \| \psi_{N,k+1} \|_{L^2}^2 + \kappa \| \nabla \psi_{N+1} \|_{L^2}^2.
\]
\begin{align*}
\leq \kappa \|\nabla \psi_{N+1}\|_{L^2}^2 + \frac{6}{\kappa} \|\tilde{u}\|_{L^2}^2 \sum_{k=1}^{N+1} k \|\psi_{N,k+1}\|_{L^2}^2.
\end{align*}

Hence, from (3.16) we get
\[
a \|\psi_{N+1}\|^2 \leq \frac{6}{\kappa} \|\tilde{u}\|_{L^2}^2 \|\psi_N\|^2
\]
and (3.7) follows. \hfill \square

The solution to (2.14) is given by
\begin{align}
\chi := \sum_{n=1}^{+\infty} \psi_n.
\end{align}

The series in (3.17) converges in the \( L^2 \) sense provided that \( q < 1 \), cf. (3.6). Also, as we shall show in Proposition \( \text{prop4141} \) below we have \( (\psi_n, \tilde{u})_{L^2} = 0 \) if \( n \) is even. From (2.15) we can write therefore that
\begin{align}
d_* = -\sum_{n=0}^{+\infty} (\psi_{2n+1}, \tilde{u})_{L^2},
\end{align}
thus, using (3.7), we conclude the following.

**Corollary 3.2.**

\begin{align}
\left| d_* + \sum_{n=0}^{M} (\psi_{2n+1}, \tilde{u})_{L^2} \right| \leq q^{2M+2}(1 - q^2)^{-1}\|\psi_1\|_{L^2}\|\tilde{u}\|_{L^2}.
\end{align}

To calculate \( \|\psi_1\|_{L^2} \) we use the spectral representation of \( \tilde{u} \)
\[
\tilde{u}(\tau_x f) = \int \tilde{u}(d\mathbf{k}; f),
\]
where \( \tilde{u}(\cdot) \) is its spectral measure that satisfies \( \tilde{u}^*(d\mathbf{k}) = \tilde{u}(-d\mathbf{k}) \) (because \( \tilde{u} \) is real valued) and
\[
\langle \tilde{u}^*(d\mathbf{k})\tilde{u}(d\mathbf{k}') \rangle = b\delta(\mathbf{k} - \mathbf{k}') \frac{E(|\mathbf{k}|)}{|\mathbf{k}|^{d-1}} \hat{f}_{11}(\mathbf{k}) d\mathbf{k} d\mathbf{k}'.
\]
Also, because \( \mathbf{x} \mapsto \tilde{u}(\tau_x f) \) is a real Gaussian field we have \( (\text{Re} \ \tilde{u}(\cdot), \text{Im} \ \tilde{u}(\cdot)) \) is jointly Gaussian. Hence,
\[
\psi_1 = (\kappa \Delta + D_0)^{-1} \tilde{u} = -\int \frac{\tilde{u}(d\mathbf{k})}{\kappa|\mathbf{k}|^2 + a}
\]
and
\begin{align}
\|\psi_1\|_{L^2}^2 = \int \int \frac{\langle \tilde{u}^*(d\mathbf{k})\tilde{u}(d\mathbf{k}') \rangle}{(\kappa|\mathbf{k}|^2 + a)(\kappa|\mathbf{k}'|^2 + a)} = b\omega_{d-1} \left( 1 - \frac{1}{d} \right) \frac{K_0}{(\kappa k^2 + a)^{2\alpha-1}}.
\end{align}
4. Calculation of \((\psi_n, \bar{u})_{L^2}\).

We start with some auxiliary notation. For any function \(F(t_1, \ldots, t_n, x_1, \ldots, x_n)\) of \(n\) temporal and spatial variables we define

\[
DF(t_1, \ldots, t_n, x_1, \ldots, x_n) := \nabla_y |_{y = 0} F(t_1, \ldots, t_n, x_1 + y, \ldots, x_n + y).
\]

\(W_n(\cdot)\) is defined inductively by

\begin{align*}
W_0(s_1, x_1) & := u(s_1, x_1), \\
W_n(s_1, \ldots, s_{n+1}, x_1, \ldots, x_{n+1}) & := u(s_{n+1}, x_{n+1}) \cdot DW_{n-1}(s_1, \ldots, s_n, x_1, \ldots, x_n).
\end{align*}

Let also \(\Delta_n := \{(s_1, \ldots, s_n) : s_1 \geq \ldots \geq s_n \geq 0\}\).

**Proposition 4.1.** We have

\[
\psi_n = -\int_{\Delta_n} \int_{(\mathbb{R}^d)^n} \int_{(\mathbb{R}^d)^n} \langle W_{n-1,1}(s, x)u_1(0,0) \rangle R_{n-1}(s, x)dsdx.
\]

Here \(s, x\) stand for the abbreviations of the ensemble of variables \(s = (s_1, \ldots, s_n), \ x = (x_1, \ldots, x_n)\), \(ds = ds_1 \ldots ds_n, \ dx = dx_1 \ldots dx_n\).

\[
R_{n-1}(s, x) := \prod_{m=1}^{n} r_n(s_m - s_{m+1}, x_m - x_{m+1}),
\]

where \(r_n(\cdot, \cdot)\) is the heat kernel defined in (2.11) and \(s_{n+1} := 0, \ x_{n+1} := 0\).

**Proof.** To show (4.3) we prove that

\[
\psi_n = -\int_{\Delta_n} \int_{(\mathbb{R}^d)^n} \int_{(\mathbb{R}^d)^n} \mathbb{E}_t[W_{n-1,1}(s, x)u_1(0,0)] R_{n-1}(s, x)dsdx,
\]

where \(\mathbb{E}_t\) denotes the conditional expectation w.r.t. \(\sigma\)-algebra \(\mathcal{U}_t\) generated by \(u(s, \cdot), \ s \leq t\).

We use an induction argument. For \(n = 1\) (4.5) is a consequence of (2.2) and the definition of the semigroup \(R(t)\).

Suppose that we have established (4.5) for a certain \(n \geq 1\). Note that

\[
\psi_{n+1} = -\int_0^{+\infty} R(t)u_n dt.
\]
Recalling the definition of $u_n$ (see \((3.3)\)) we get
\[
    u_n = -\int \cdots \int \int \cdots \int_{\Delta_n} u(0,0) \cdot \mathbb{E}_0[D W_{n-1,1}(s,x)] R_{n-1}(s,x) ds dx
\]
and in consequence
\[
    R(t) u_n = -\int \cdots \int \int \cdots \int_{\Delta_n} u(t,z) \cdot \mathbb{E}_0[D W_{n-1,1}(s+t,x)] R_{n-1}(s,x) r_n(t,z) ds dx dz.
\]
Here $s + t := (s_1 + t, \ldots, s_n + t), x + z := (x_1 + z, \ldots, x_n + z)$. Setting $s'_{n+1} := t$, $s'_{i} := s_i + t$, $x'_{n+1} := z$, $x'_{i} := x_i$, $i = 1, \ldots, n$ it is clear from \((4.6)\) and definitions \((4.2), (4.4)\) that formula \((4.3)\) holds for $n + 1$.

\[\text{Remark 4.2.}\] From \((4.3)\) and elementary properties of Gaussian variables we conclude that $(\psi_{2n}, \tilde{u})_{L^2} = 0$ for all $n$.

To calculate $(W_{2n,1}(s,x) u_1(0,0))$ appearing in the formula for $(\psi_{2n+1}, \tilde{u})_{L^2}$, cf. Proposition \[\text{prop414}\], we need to compute the mathematical expectation of a multiple product of Gaussian random variables, cf. \[\text{janson}\]. For that purpose it is convenient to use a graphical representation, borrowed from quantum field theory. We refer to, e.g. \[\text{janson}\]. A \textit{Feynman diagram} $G$ (of order $n \geq 0$ and rank $r \geq 0$) is a graph consisting of a set $B(G)$ of $n$ vertices, that are positive integers, and a set $E(G)$ of $r$ edges without common endpoints. So there are $r$ pairs of vertices, each joined by an edge, and $n - 2r$ unpaired vertices, called \textit{free vertices}. An edge whose endpoints are $m, n \in B$ is represented by $\tilde{m}n$ (unless otherwise specified, we always assume $m < n$). A diagram $G$ is said to be \textit{based on} $B(G)$. Denote the set of free vertices by $A(G)$, so $A(G) = F \setminus E(G)$. The diagram is \textit{complete} if $A(G)$ is empty and \textit{incomplete}, otherwise.

Let $Z_n := \{1, \ldots, n\}$. Denote by $\mathcal{G}_n$ the family of all complete Feynman diagrams based on $Z_n$ ($n$ must be then even). For a given $G \in \mathcal{G}_n$ and any $l \leq n$ we denote by $V_l(G)$ all those vertices $m \leq l$ for which there is $n > l$ such that $\tilde{m}n \in E(G)$.

Suppose that $G, G'$ are Feynman diagrams based on $Z_n, Z_{n-1}$ respectively. We call $G'$ an \textit{immediate predecessor} of $G$ and denote this $G' \hookrightarrow G$ if $E(G') \subseteq E(G)$. Diagram $G$ is called \textit{admissible} if $A(G) \setminus \{n\} \neq \emptyset$. For $n \geq 1$ we define a class $\mathcal{G}_n$ that consists of all sequences $F := (F_k)_{k=1}^n$ of Feynman diagrams $F_1 \hookrightarrow F_2 \hookrightarrow F_n$, such that each $F_k$, based on $Z_k$, is admissible. For $n$ even we denote by $\mathcal{G}_n^0$ the class of those sequences $F_1 \hookrightarrow F_2 \hookrightarrow F_n$ for which $(F_k)_{k=1}^{n-1} \in \mathcal{G}_{n-1}$ and $F_n \in \mathcal{G}_n$. Let $F \in \mathcal{G}_n$, or $\mathcal{G}_n^0$. Denote by $A_k(F) := A(F_k)$,
Lemma 4.3. Let \( n \geq 0 \) and \( s = (s_1, \ldots, s_{2n+1}) \in \Delta_{2n+1} \), \( x = (x_1, \ldots, x_{2n+1}) \), \( k = (k_1, \ldots, k_{2n+1}) \). We have then

\[
(W_{2n,1}(s, x)u_1(0, 0)) = (-1)^n b^{n+1} \int \ldots \int \exp \left\{ i \sum_{m=1}^{2n+1} k_m \cdot x_m \right\} P_j(s, k; F)Q_{j}(k; F_{2n+2}) dk. \tag{4.7}
\]

The summation extends over all integer valued multi-indices \( j = (j_1, \ldots, j_{2n+2}) \) of length \( 2n + 2 \), such that \( j_1 = j_{2n+2} = 1 \) and all diagrams \( F \in \mathcal{F}_{2n+2} \). Also for a given \( F \)

\[
\times \prod_{l=1}^{2n} \left\{ \sum_{m \in A_l(F)} \exp \left\{ -aa_l(F)(s_l - s_{l+1}) \right\} \left[ 1 - \exp \left\{ -2a(s_l - s_{l+1}) \right\} \right]^{c_l(F)} \right\}, \tag{4.8}
\]

\[
Q_j(k; F_{2n+2}) = \prod_{m \in A_l(F)} \mathcal{E}(|k_m|) |k_m|^{d_l-1} \, \hat{f}_{j, m, j_m'}(k_m) \delta(k_m + k_m'). \tag{4.9}
\]

Here, for abbreviation sake \( dk := dk_1 \ldots dk_{2n+2} \).

This lemma follows from an analogous argument as used in the proof of Lemma 1 of [4]. For reader’s convenience we present it in Appendix A below.

Using Proposition 4.1 and Lemma 4.3 contained in the following section we have the following formula.

Proposition 4.4. We have

\[
(\psi_{2n+1}, \hat{u})_{L^2} = (-1)^n b^{n+1} (2a)^{-2n+1} \sum_{l=1}^{2n+1} c_l(F)! \int \ldots \int \prod_{l=1}^{2n} \left( \sum_{m \in V_l(F_{2n+2})} k_m(j_{l+1}) \right)
\]

\[
\times \prod_{l=1}^{2n+1} \left[ \frac{c_l(F)}{2} a_l(F) + p + \frac{\kappa}{2a} \sum_{m \in V_l(F_{2n+2})} k_m^2 \right]^{-1} Q_j(k_1, \ldots, k_{2n+2}; F_{2n+2}) dk. \tag{4.10}
\]

The summation range is the same as in Lemma 4.3.
Proof. Using Fourier transform we can write
\[ R_{2n}(s, x) = \int \cdots \int \prod_{m=1}^{2n+1} \exp \left\{-\kappa |q_m|^2(s_m - s_{m+1}) + iq_m \cdot (x_m - x_{m+1})\right\} dq, \]
with \( dq = dq_1 \ldots dq_{n+1} \). Using formula (4.3) to represent \((\psi_{2n+1}, \tilde{u})_{L^2}\) and substituting from Lemma 1.3 for \( (W_{2n,1}(s, x)u_1(0, 0)) \) we obtain that

\[ (4.11) \quad (\psi_{2n+1}, \tilde{u})_{L^2} = (-1)^{n+1} b^{n+1} \sum \int \cdots \int \prod_{m=1}^{2n+1} \exp \left\{-\kappa |q_m|^2(s_m - s_{m+1})\right\} \]
\[ \times \prod_{m=1}^{2n+1} \exp \left\{i(k_m + q_m - q_{m-1}) \cdot x_m \right\} P_j(k_1, \ldots, k_{2n+1}; F)Q_j(k_1, \ldots, k_{2n+2}; F_{2n+2}) ds dx dk dq, \]

here \( q_0 := 0 \) and and the summation range is the same as in Lemma 1.3. Performing the integration over \( x \) variables yields the expression \( \prod_{m=1}^{2n+1} \delta(k_m + q_m - q_{m-1}) \), which in turn implies that \( q_1 = -k_1, q_m = q_{m-1} - k_m \). Hence \( q_l = -\sum_{m=1}^{l} k_m \). Changing variables \( s_m' := s_m - s_{m+1}, m = 1, \ldots, 2n+1 \) \((s_{2n+2} = 0)\) and substituting for \( P_j(k_1, \ldots, k_{2n+1}; F) \) from (1.8) we get

\[ (\psi_{2n+1}, \tilde{u})_{L^2} = (-1)^{n+1} b^{n+1} \sum \int \cdots \int \prod_{l=1}^{2n} \left( \sum_{m \in A_l(F)} k_{m,j_l+1} \right) P_j(k_1, \ldots, k_{2n+2}; F_{2n+2}) \]
\[ \times \prod_{l=1}^{2n+1} \left[ \int_0^{+\infty} \exp \left\{-\left(aq_l(F) + \kappa \sum_{m=1}^{l} k_m \right)^2 \right\} s_l' \right\left(1 - e^{-2as_l'}\right) ds_l' \] \[ \left.e_l(F) \right] dk. \]

Substituting \( t_l := e^{-2as_l'} \) in the integral w.r.t. \( s_l' \) we conclude that

\[ (\psi_{2n+1}, \tilde{u})_{L^2} = (-1)^{n+1} b^{n+1}(2a)^{-2n-1} \sum \int \cdots \int \prod_{l=1}^{2n} \left( \sum_{m \in A_l(F)} k_{m,j_l+1} \right) Q_j(k_1, \ldots, k_{2n+2}; F_{2n+2}) \]
\[ \times \prod_{l=1}^{2n+1} B \left( \frac{1}{2} \delta_l(F) + \kappa \sum_{m=1}^{l} k_m \right)^2, e_l(F) + 1 \] \[ dk. \]

Here \( B(\cdot, \cdot) \) denotes the Euler beta function. Note that for a fixed \( l \leq 2n+1 \) for any \( m, m' \leq l \) such that \( mm' \in E(F_{2n+2}) \) we necessarily have \( k_m + k_{m'} = 0 \) thus the summation range in the sum of \( k \)-s reduces to \( V_l(F_{2n+2}) \). Using the classical formula

\[ B(x, n+1) = n! \left[ \prod_{p=0}^{n} (x + p) \right]^{-1} \]
valid for any positive integer \( n \) we conclude (1.10). \( \square \)
Computations of the terms appearing in $\mathcal{E}_2$. For $n = 0$ the class $\mathcal{E}_2$ consists of a single diagram sequence $1 \rightarrow \hat{1}2$. From (4.10), after a simple calculation, we obtain

$$\omega_1 \cdot \bar{u}\nu \in \mathcal{L}_2 = -b \int_{|k| \leq K_0} \left(1 - \frac{k_1^2}{|k|^2}\right) \frac{1}{a + \kappa|k|^2} \times \frac{dk}{|k|^{2\alpha + d - 2}}$$

$$\omega_1 \cdot \bar{u}\nu \in \mathcal{L}_2 = -\omega_1 \cdot \bar{u}\nu \in \mathcal{L}_2$$

The second equality in (4.10) is a consequence of isotropy. For $n = 1$ there are only four sequences of diagrams from $\mathcal{E}_4$ for which the corresponding terms of (4.10) are non vanishing. They are A) $1 \rightarrow 12 \rightarrow 123 \rightarrow \hat{1}4 \hat{2} \hat{3} 23$, B) $1 \rightarrow 12 \rightarrow 123 \rightarrow \hat{1}3 \hat{2} \hat{4} 23$, C) $1 \rightarrow 12 \rightarrow \hat{1}3 \hat{2} \hat{4} 23, D) 1 \rightarrow 12 \rightarrow \hat{2}3 \hat{1} \rightarrow \hat{2}3 \hat{4} 1$. The corresponding terms of the sum appearing on the right hand side of (4.10), after using isotropy, can be calculated as follows equal

$$A = \frac{b^2}{d} \int \int \frac{1}{|k_1||k_2|^2} \left[1 - \left(\frac{k_1 \cdot k_2}{|k_1||k_2|}\right)^2\right]$$

$$B = \frac{b^2}{d} \int \int \frac{1}{|k_1||k_2|^2} \left[1 - \left(\frac{k_1 \cdot k_2}{|k_1||k_2|}\right)^2\right]$$

$$C = \frac{2ab^2}{d} \int \int \frac{1}{|k_1||k_2|^2} \left[1 - \left(\frac{k_1 \cdot k_2}{|k_1||k_2|}\right)^2\right]$$

$$D = \frac{2ab^2}{d} \int \int \frac{1}{|k_1||k_2|^2} \left[1 - \left(\frac{k_1 \cdot k_2}{|k_1||k_2|}\right)^2\right]$$
Adding all these terms we conclude that
\[
(\psi_3, \tilde{u})_{L^2} = \frac{b^2}{d} \int \frac{1}{|k_1||k_2|} \left[ 1 - \left( \frac{k_1 \cdot k_2}{|k_1||k_2|} \right)^2 \right] \left[ \frac{(d-1)|k_1|^2}{a + \kappa|k_1|^2} - \frac{k_1 \cdot k_2}{a + \kappa|k_2|^2} \right] d\kappa_1 d\kappa_2
\]
\[
\times \frac{1}{a + \kappa|k_1|^2} \times \frac{1}{2a + \kappa|k_1| + |k_2|^2} \times \frac{1}{(|k_1||k_2|)^{2a + d - 2}}.
\]

A simple calculation shows that the approximation of the eddy diffusivity obtained by using (5.19), with \( M = 1 \), has an error bounded by \( 0.1 \| \tilde{u} \|_{L^2} \| u \|_{L^2} \| \psi_1 \|_{L^2} \) provided that \( a \kappa \approx 22 \).

To find a more explicit formula for the terms appearing in (5.10) when \( n \geq 2 \) we fix a sequence of Feynman diagrams \( \mathcal{F} \in \mathcal{G}_{2n+2} \) and denote by \( I(\mathcal{F}) \) the corresponding term appearing on the right hand side of (5.10). Let us consider two cases.

Case 1), when bond \( 1, 2n + 2 \in E(\mathcal{F}_{2n+2}) \). Let \( \tilde{m} \tilde{m}' \in E(\mathcal{F}_{2n+2}) \) be such that \( 1 < m < m' < 2n + 2 \). Performing the summation over the respective multi-indices we get
\[
\sum_{j_m, j_{m'}} \left( \sum_{p \in V_{m-1}(\mathcal{F}_{2n+2})} k_{p, j_m} \right) \left( \sum_{p' \in V_{m'-1}(\mathcal{F}_{2n+2})} k_{p', j_{m'}} \right) \Gamma_{j_{m}, j_{m'}}(k_m)
\]
\[
= \sum_{p \in V_{m-1}(\mathcal{F}_{2n+2}), p' \in V_{m'-1}(\mathcal{F}_{2n+2})} \left( k_p \cdot k_{p'} - \frac{(k_p \cdot k_m)(k_{p'} \cdot k_m)}{|k_m|^2} \right).
\]

Denote by \( m(p), m'(p), p = 1, \ldots, n + 1 \) all the edges of \( \tilde{m} \tilde{m}' \in E(\mathcal{F}_{2n+2}) \) whose left endpoints are enumerated in increasing order. We can write then that

\[
I(\mathcal{F}) = \frac{(-b)^{n+1}}{(2a)^{2n+1}} \int_{|k_1|, \ldots, |k_{n+1}| \leq K_0} \left( 1 - \frac{k_{11}^2}{|k_1|^2} \right) \mathcal{J}(\mathcal{F}_{2n+2})
\]
\[
\times \prod_{l=1}^{2n+1} \left[ e_l(\mathcal{F})! \prod_{p=0}^{e_l(\mathcal{F})} \left( \frac{1}{2} a_l(\mathcal{F}) + p + \frac{\kappa}{2a} \sum_{m(p) \in V_l(\mathcal{F}_{2n+2})} k_p \right) \right]^{-1} \frac{d\kappa_1 \ldots d\kappa_{n+1}}{(|k_1| \ldots |k_{n+1}|)^{2a + d - 2}}.
\]

Here for a given complete Feynman diagram \( \mathcal{G} \in \mathcal{G}_{2n+2} \) such that \( 1, 2n + 2 \in E(\mathcal{G}) \) we set

\[
\mathcal{J}(\mathcal{G}) := \prod_{l=1}^{n+1} \sum_{m(p) \in V_{m(l)-1}(\mathcal{F}_{2n+2}), m(p') \in V_{m(l')-1}(\mathcal{F}_{2n+2})} \left( k_p \cdot k_{p'} - \frac{(k_p \cdot k_l)(k_{p'} \cdot k_l)}{|k_l|^2} \right)
\]

As above \( 1 = m(1) < \ldots < m(n+1) \) denote the left vertices of all the edges from \( E(\mathcal{G}) \).
Using isotropy we can further simplify the formula and obtain that

\[
I(\mathcal{F}) = \left(1 - \frac{1}{d}\right) \frac{(-b)^{n+1}}{(2\alpha)^{2n+1}} \int \cdots \int |k_1|, \ldots, |k_{n+1}| \leq K_0 \mathcal{J}(\mathcal{F}_{2n+2})
\]

\[
\times \prod_{l=1}^{2n+1} e_l(\mathcal{F})! \prod_{p=0}^{e_l(\mathcal{F})} \left(\frac{1}{2} a_l(\mathcal{F}) + p + \frac{\kappa}{2\alpha} \sum_{m(p) \in \mathcal{V}_l(\mathcal{F}_{2n+2})} |k_p| \right)^2 - 1 \frac{dk_1 \cdots dk_{n+1}}{(|k_1| \cdots |k_{n+1}|)^{2^{n+1}d-2}}.
\]

**Case 2)**, when bond 1, 2n + 2 \( \notin E(\mathcal{F}_{2n+2}) \). Let \( q > 1 \) be such that \( m(q), 2n + 2 \in E(\mathcal{F}_{2n+2}) \). After a similar calculation to the one done in the previous case we obtain that

\[
I(\mathcal{F}) = \frac{(-b)^{n+1}}{d(2\alpha)^{2n+1}} \int \cdots \int |k_1|, \ldots, |k_{n+1}| \leq K_0 \mathcal{J}(\mathcal{F}_{2n+2})
\]

\[
\sum_{p,p'} \left[ k_p \cdot k_{p'} + \frac{(k_q \cdot k_{p'}) (k_1 \cdot k_p) (k_1 \cdot k_q)}{|k_1| |k_q|} - \frac{(k_p \cdot k_q) (k_{p'} \cdot k_q)}{|k_q|^2} - \frac{(k_p \cdot k_1) (k_{p'} \cdot k_1)}{|k_1|^2} \right]
\]

\[
\times \prod_{l=1}^{2n+1} e_l(\mathcal{F})! \prod_{p=0}^{e_l(\mathcal{F})} \left(\frac{1}{2} a_l(\mathcal{F}) + p + \frac{\kappa}{2\alpha} \sum_{m(p) \in \mathcal{V}_l(\mathcal{F}_{2n+2})} |k_p| \right)^2 - 1 \frac{dk_1 \cdots dk_{n+1}}{(|k_1| \cdots |k_{n+1}|)^{2^{n+1}d-2}}.
\]

Here for a given \( G \in \mathcal{S}_{2n+2} \) such that \( m(q), 2n + 2 \in E(G) \) for some \( q > 1 \) we set

\[
\mathcal{J}(G) := \prod_{l \notin \{1, q\}} \sum_{m(p) \in \mathcal{V}_m(1_{-1}(G))} \left( k_p \cdot k_{p'} - \frac{(k_p \cdot k_q) (k_{p'} \cdot k_q)}{|k_q|^2} \right)
\]

The range of the sum in the upper line of the above equality is the same as in \( \text{ex}^{10} \), the range of the sum in the middle line extends over \( p, p' \) such that \( m(p) \in \mathcal{V}_{m(1)}(\mathcal{F}_{2n+2}), m(p') \in \mathcal{V}_{m(q)-1}(\mathcal{F}_{2n+2}) \).

**APPENDIX A. THE PROOF OF LEMMA 4.3**

Using the Fourier transform in the spatial variable we can write that

\[
\hat{u}(t, x) = \int e^{ix\cdot k} \hat{u}(t, dk),
\]

where \( \hat{u}(t, dk) = (\hat{u}_1(t, dk), \ldots, \hat{u}_d(t, dk)) \) is a real-valued Gaussian spectral measure with the structure function

\[
\langle \hat{u}^*(t, dk) \otimes \hat{u}(s, dk') \rangle = be^{-a|t-s|} \left( \frac{\delta \Gamma(k)}{|k|^{d-1}} \right) \delta(k - k') dk dk'.
\]

satisfying \( \hat{u}^*(t, dk) = \hat{u}(t, -dk) \).
To show the lemma it suffices only to prove that for arbitrary \( s = (s_1, \ldots, s_{n+1}) \in \Delta_{n+1}, \)
\( s_{n+2} \leq s_{n+1} \) we have

\[
\mathbb{E}_{s_{n+2}} W_{n,1}(s, x) = i^n \sum \int \ldots \int b^{f_{n+1}(F)} \exp \left\{ i \sum_{m=1}^{n+1} k_m \cdot x_m \right\} \hat{P}_j(s, k; s_{n+2}, F) \hat{Q}_j(dk; s_{n+2}, F).
\]

Here \( f_m(F) = \sum_{l \leq m} c_l(F) \) denotes the cardinality of \( E(F_m) \). The summation extends over all integer valued multi-indices \( j = (j_1, \ldots, j_{n+1}) \), such that \( j_1 = 1 \) and all Feynman diagrams \( F \in \mathcal{S}_{n+1} \). As we recall, \( \mathbb{E}_t \) is the conditional expectation w.r.t. the \( \sigma \)-algebra \( \mathcal{U}_t \) generated by \( u(s, \cdot), s \leq t \) and

\[
\mathbb{E}_{s_{n+2}} W_{n,1}(s, x) = \mathbb{E}_{s_{n+3}} \left[ u(s_{n+2}, x_{n+2}) \cdot D \mathbb{E}_{s_{n+2}} W_{n,1}(s_1, \ldots, s_{n+1}, x_1, \ldots, x_{n+1}) \right].
\]

Calculate \( \mathbb{E}_{s_{n+2}} W_{n,1}(\cdot) \) using (A.3). Note that

\[
\mathbb{E}_{s_{n+2}} W_{n+1,1}(s, x) = \mathbb{E}_{s_{n+3}} \left[ u(s_{n+2}, x_{n+2}) \cdot D \mathbb{E}_{s_{n+2}} W_{n,1}(s_1, \ldots, s_{n+1}, x_1, \ldots, x_{n+1}) \right].
\]

The summation extends over all integer valued multi-indices \( j = (j_1, \ldots, j_{n+2}) \), such that \( j_1 = 1 \) and all Feynman diagrams \( F \in \mathcal{S}_{n+1} \). Differentiating w.r.t. \( y_{j_{n+2}} \) and using the identification rule \( k_m = -k_{m'} \), when \( m m' \in E_{n+1}(F) \) we obtain that the right hand side of (A.6) equals

\[
\mathbb{E}_{s_{n+3}} \left[ \hat{u}_{j_{n+2}}(s_{n+2}, k_{n+2}) \frac{\partial}{\partial y_{j_{n+2}}} \bigg|_{y=0} \exp \left\{ i \sum_{m=1}^{n+1} k_m \cdot y \right\} \hat{Q}_j(dk; s_{n+2}, F) \right].
\]
\[
\left( \sum_{m \in A_{n+1}(\mathcal{F})} k_{m,j+n+2} \right) E_{s_{n+3}} \left[ \hat{u}_{j+n+2}(s_{n+2}, k_{n+2}) \hat{Q}_j(dk; s_{n+2}, \mathcal{F}) \right].
\]

From elementary properties of Gaussian variables we conclude that
\[
\hat{u}_j(s_{n+2}, dk; s_{n+3}) := E_{s_{n+3}} \hat{u}_j(s_{n+2}, dk) = e^{-a(s_{n+2}-s_{n+3})} \hat{u}_j(s_{n+3}, dk).
\]

The conditional expectation
\[
\hat{u}_j^+(s_{n+2}, dk; s_{n+3}) := \hat{u}_j(s_{n+2}, dk) - E_{s_{n+3}} \hat{u}_j(s_{n+2}, dk)
\]
is independent of the \(\sigma\)-algebra \(\mathcal{U}_{s_{n+2}}\). Moreover, note that
\[
\langle \hat{u}_j^+(s_{n+2}, dk; s_{n+3}) \hat{u}_j^+(s_{n+2}, dk'; s_{n+3}) \rangle = \left[ 1 - e^{-2a(s_{n+2}-s_{n+3})} \right] \langle \hat{u}_j(0, dk) \hat{u}_j(0, dk') \rangle.
\]

Replace each \(\hat{u}_{jm}(s_{n+2}, dk_m)\) appearing in (A.7) i.e. \(\hat{u}_{j+n+2}(s_{n+2}, k_{n+2})\) and the spectral measures that occur in \(\hat{Q}_j(dk; s_{n+2}, \mathcal{F})\), by
\[
e^{-a(s_{n+2}-s_{n+3})} \hat{u}_{jm}(s_{n+3}, dk_m) + \hat{u}_{jm}^+(s_{n+2}, dk_m; s_{n+3}).
\]

The conditional expectation \(E_{s_{n+3}}\) can be expressed using the expectation of the product of the terms in the form \(\hat{u}_{jm}^+(s_{n+2}, dk_m; s_{n+3})\) times the product of \(e^{-a(s_{n+2}-s_{n+3})} \hat{u}_{jm}(s_{n+3}, dk_m)\).

In order to finish the induction argument we apply the rules of calculating the expectation of products of Gaussian random variables using Feynman diagrams. To prove that out of the diagrams generated in that way we only need to take into account those belonging to \(\mathcal{S}_{n+2}\) it suffices only to show that

\[
(A.8) \quad i^{n+1} \sum \int \ldots \int b^{f_{n+1}(\mathcal{F})} \exp \left\{ i \sum_{m=1}^{n+2} k_m \cdot x_m \right\} \hat{P}_j(s, k; s_{n+2}, \mathcal{F})
\]

\[
\left( \sum_{m \in A_{n+1}(\mathcal{F})} k_{m,j+n+2} \right) \left\langle \hat{u}_{j+n+2}^+(s_{n+2}, k_{n+2}; s_{n+3}) \hat{Q}_j^+(dk; s_{n+2}, s_{n+3}, \mathcal{F}) \right\rangle = 0
\]

and

\[
(A.9) \quad i^{n+1} \sum \int \ldots \int b^{f_{n+1}(\mathcal{F})} \exp \left\{ i \sum_{m=1}^{n+2} k_m \cdot x_m \right\} \hat{P}_j(s, k; s_{n+2}, \mathcal{F})
\]

\[
\left( \sum_{m \in A_{n+1}(\mathcal{F})} k_{m,j+n+2} \right) e^{-a(s_{n+2}-s_{n+3})} \hat{u}_j(s_{n+3}, dk_{n+2}) \left\langle \hat{Q}_j^+(dk; s_{n+2}, s_{n+3}, \mathcal{F}) \right\rangle = 0.
\]

Here
\[
\hat{Q}_j^+(dk; s_{n+2}, s_{n+3}, \mathcal{F})
\]
\[ := \prod_{m,n' \in E_{n+1}(\mathcal{F})} \mathcal{E}(\{k_m\}) \cdot \Gamma_{j_m,j_{m'}}(k_m) \delta(k_m + k_{m'}) \, dk_m \, dk_{m'} \prod_{m \in A_{n+1}(\mathcal{F})} \hat{u}^\perp_{j_m}(s_{n+2}, d k_m; s_{n+3}). \]

Equality in (50901) is a direct result of spatial homogeneity of the velocity field. As for (50818), note that since \[ \sum_{j_m+j_{m'}=n+2} \hat{u}_{j_{m'+2}}^\perp (s_{n+2}, d k_{n+2}; s_{n+3}) = 0 \] the expression appearing on its left hand side equals

\[ \int_\cdot \int_\cdot \exp \left\{ \sum_{m=1}^{n+2} \nu \cdot (x_m + y) \right\} \hat{Q}_{j_m}^\perp (d k; s_{n+2}, s_{n+3}, \mathcal{F}) \quad = \quad 0. \]

The last equality in (A.10) is a consequence of spatial stationarity of the field.

References


Institute of Mathematics Polish Academy of Sciences, Warsaw
Institute of Mathematics, UMCS, Lublin

E-mail address: komorow@hektor.umcs.lublin.pl

Institute of Mathematics, UMCS, Lublin

E-mail address: pwidel@golem.umcs.lublin.pl